ON A METHOD OF CONSTRUCTING OPTIMAL TRAJECTORIES

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ON A METHOD OF CONSTRUCTING OPTIMAL TRAJECTORIES

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The present article describes a method of determining optimal trajectories (with respect to the duration of the transient process) for linear nonstationary systems in the case of limited control function moduli.

Let us consider the system described by the differential equations

$$\frac{dx}{dt} = P(t)x + b\eta + \varphi(t), \qquad (0.1)$$

where $x = \{x_i\}$ is an n-dimensional vector with coordinates x_i of representing point x in the phase space of the system, $P(t)$ is the n X n-matrix whose elements $p_{ij}(t)$ are continuous functions of time $t \in (0, \infty)$, $b = (b_1, \ldots, b_n)$ is a constant vector, $\varphi(t) = \{\varphi_1(t), \ldots, \varphi_n(t)\}$ is a continuous vector function describing the external signal, and $\eta(t)$ is a scalar control magnitude.

Let us formulate two optimal regulation problems:

Problem A. For given $x = x_0$, $t = t_0 = 0$ to determine the optimal control in the form of a piecewise-smooth function $\eta^0(t)$ such that

$$|\eta(t)| \leq 1 \qquad (0.2)$$

and trajectory $x(x_0, t, \eta^0)$ of system (0.1) reaches point $x = 0$ in the minimum time $t = T^0$ (or $\eta = \eta^0(t)$).

Note. By making the substitution $y = x - x(t)$, the problem of finding an optimal control from the point $x = x_0$

to a point moving along the curve $x = f(t)$, where $f(t)$ is a continuously differentiable vector function.

Problem B. For given $x = x_0$, $t = t_0 = 0$, we are to determine the optimal control $\eta^0(t)$ satisfying condition (0.2) such that the trajectory $x(x_0, t, \eta^0, M)$ of system (0.1) is traced out on the given manifold M of space in the shortest possible time $t = T^0$.

The problems of optimal control have occupied many writers. A formulation of the problem and the first results will be found in the writings of A.A. Fel'dbaum (ref. 1). The most profound results in the mathematical theory of optimal processes have so far been achieved by L.S. Pontryagin, V.G. Boltyanskiy, and R.V. Gamkrelidze. They have considered optimal processes described both by general systems of differential equations

$$\frac{dx}{dt} = f(x, u)$$

where $u = \{u_1, \ldots, u_r\}$ is a control vector and by linear systems. Some of these results are presented in article 2 in the bibliography. In the case of linear systems, important results have been achieved by R.V. Gamkrelidze (refs. 3 and 4). Mention must also be made of the contributions of R. Bellman, J. Glicksberg, O. Gross (ref. 5) and J. La Salle (ref. 6). The problems involved in the theory of optimal control are dealt with in the writings of L.I. Rozonoer (refs. 7 and 8), which take as the point of departure L.S. Pontryagin's Maximum Principle. Extremely general optimal control existence theorem have recently been proved by A.F. Fillipov (ref. 9).

In constructing actual concrete optimal trajectories there arises a certain difficulty which consists in the following: the optimum conditions make it possible to calculate the optimal trajectory which emerges from point $x = x_0$ by integrating a certain system of equations (the Hamiltonian system given in ref. 2) and operating with a system of n arbitrary constants; there now arises the problem of determining these constants in such a way that the trajectory passes precisely into point $x = 0$. This last problem presents particular difficulties in the presence of disturbance functions $\varphi(t)$ (or when there is a need to bring the trajectory onto curve $x = f(t)$. Analogous difficulties arise, moreover, in solving problem B.

The purpose of the present article is to describe one method of constructing optimal trajectories which in a number of cases makes it possible to circumvent the difficulties

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presented by boundary problems. This method consists in approximating problems A and B (with limitation (0.2) taken into account, which have discontinuous optimal controls with problems having smoother solutions, as well as in introducing parameters Θ, μ into auxiliary systems of equations [see note]. The determination of optimal trajectories is thus reduced to the integration of ordinary differential equations describing the dependence of optimal solutions on parameters Θ, μ and applying a set of known initial conditions. [Note: The dependence of solutions to optimal problems on parameters has been studied by F.M. Kirillov (ref. 13); his results are used in the present article].

It should be noted that these equations are rather cumbersome, so that integration is possible only by numerical methods; for this reason, the method described may turn out to be useful for determining individual optimal trajectories, but not for the solution of systems synthesis problems.

1. In this paragraph we shall define certain auxiliary notions and symbols.

Along with system (0.1), we shall examine the auxiliary system of equations

$$\frac{dx}{dt} = \Theta P(t)x + B(\mu)u + \Theta \varphi(t) \qquad (1.1)$$

where μ, Θ are parameters $(0 \leqslant \Theta \leqslant 1; 0 \leqslant \mu \leqslant 1)$, $B(\mu)$ is an $n \times n$-dimensional matrix with coefficients given by the formulas

$$b_{i1} = b_i \quad (i = 1,\ldots,n) \qquad (1.2)$$

$$b_{ij} = \varepsilon_{ij}\mu \quad (i = 1,\ldots,n; \; j = 2,\ldots,n) \qquad (1.3)$$

(ε_{ij} are constants such that for $\mu \in (0,1]$ matrix $B(\mu)$ is non-singular), $u = \{u_1,\ldots,u_n\}$ is the control vector function. For values $\Theta \in [0,1]$, $\mu \in (0, 1]$, problems $A_{\Theta\mu}$ and $B_{\Theta\mu}$ will be considered in comparison to problems A and B.

Problem $A_{\Theta\mu}$. For $x = x_0$, $t = t_0 = 0$ to find the optimal trajectory $u^0(t, \Theta, \mu)$ which satisfies the condition

$$u_1^2(t) + \ldots + u_n^2(t) \leqslant 1 \qquad (1.4)$$

and is such that the trajectory $x(x_0, t, u^0)$ of system (1.1) reaches point $x = 0$ in the shortest time $t = T^0(\Theta, \mu)$ for

3.

Problem $B_{\Theta,\mu}$. For $x = x_0$, $t = t_0 = 0$ to find the optimal control $u^0(t, \Theta, \mu, M)$ which satisfies condition (1.4) and is such that the trajectory $x(x_0, t, u^0, M)$ of system (1.1) passes onto manifold M of space x in the shortest time $t = T^0(\Theta, \mu, M)$.

Note 1. In problem $B_{\Theta\mu}$ manifold M can also be considered as dependent on parameters Θ and μ; we shall limit ourselves here, however, to the case where M is assumed to be independent of parameters Θ and μ.

Note 2. To simplify the equations derived below, the factor Θ appearing in front of function $\varphi(t)$ in system (1.1) also need not be derived. This, however, will complicate the solution of problem $A_{01}(B_{01})$ somewhat.

Let us introduce the following notation:
$F(t, \Theta)$ is the fundamental matrix of solutions to the system

$$\frac{dx}{dt} = \Theta P(t)x \qquad (1.5)$$

$$(F(0, \Theta) = E);$$

$F^{-1}(t, \Theta)$ is the inverse of matrix F;
$H(t, \Theta, \mu)$ is the matrix

$$H(t, \Theta, \mu) = F^{-1}(t, \Theta)B(\mu). \qquad (1.6)$$

Let us define the matrices $R_k(t)$ by the recursion formula

$$R_1 = E, \quad R_{k+1} = \frac{dR_k}{dt} - R_k P \quad (k = 1, \ldots, n-1) \qquad (1.7)$$

and assume that vectors $R_k b$ $(k = 1, \ldots, n)$ are linearly independent for $t \geqslant 0$ with the possible exception of certain isolated values of t_ℓ, i.e.,

$$\sum_{k=1}^{n} \lambda_k R_k b \neq 0 \qquad (1.8)$$

$$\left(\sum_{k=1}^{n} \lambda_k^2 \neq 0, \ t \neq t_\ell, \ \ell = 1, 2, \ldots \right).$$

For the linear stationary case where $P(t) = A = $ const, condition (1.8) is given in (ref. 3). Under conditions (1.8)

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functions $h_{i1}(t) = \{H\}_{i1}$, the elements of matrix (1.6) satisfy the condition

$$\sum_{i=1}^{n} \lambda_i h_{i1}(t) \neq 0 \quad \left(\sum_{i=1}^{n} \lambda_i^2 \neq 0 \right) \tag{1.9}$$

for all $t \leq 0$ with the possible exception of certain isolated values $t = t_m^*$. This fact is checked by differentiating the expression $\beta(t) = \sum \lambda_i h_{i1}(t)$ with respect to t $\quad n-1$ times. Derivatives $\dfrac{d^k \beta}{dt^k}$ are expressed in terms of vectors $B_{k+1} b$ (see ref. 10). From the fact of the linear independence of these vectors it follows that for all $t \leq 0$ (with the exception of isolated values) we cannot simultaneously have

$\beta = 0, \ldots, \dfrac{d^{n-1}\beta}{dt^{n-1}} = 0$; this proves (1.9).

Henceforth, the initial point $x = x_0$ will be considered fixed and such that there exists a permissible control $\eta(t)$ (corresponding to $u(t)0$ which satisfies conditions (0.2) (or (1.4) respectively) and generates a trajectory reaching point $x = 0$ (on manifold M) in a finite time T. The problem of evaluating region G whose points $x_0 \in G$ satisfy this condition is not germane to the present article. Let us merely note that these conditions are satisfied for all points x_0 for a sufficiently broad class of properly stable linear systems for $\psi = 0$. We shall henceforth assume that $x_0 \in G$ for all problems $A_{\theta\mu}$, $B_{\theta\mu}$.

2. This paragraph deals with problem A and the transition from problem A to problem $A_{\theta\mu}$.

In ref. 11 it is shown that under conditions (1.8) problem A has a single solution, while the optimal control time T^0 is determined from the condition that T^0 is the least root of the equation

$$\lambda(T) = \min \int_0^T \left| \sum_{i=1}^{n} \lambda_i h_{i1}(\tau) \right| d\tau = 1 \tag{2.1}$$

$$\left(\sum_{i=1}^{n} \lambda_i c_i = 1 \right)$$

where c_i are components of vector c, defined by the equation

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$$c = -x_0 - \int_0^T \Theta\, r^{-1}(\tau, \Theta)\varphi(\tau)d\tau \qquad (2.2)$$

$$(\Theta = 1).$$

The optimal control $\eta^0(t)$ is defined by the formula*

$$\eta^0(t) = \text{sign}\left(\sum_{i=1}^n \lambda_i^0 h_{i1}(t)\right), \qquad (2.3)$$

where λ_i^0 are solutions to problem (2.1).

Note. In case $\varphi \equiv 0$, vector $c = -x_0 = \text{const}$, $\lambda(T)$ is a monotonically increasing function T and equation (2.1) has a single root.

If problem $A_{\Theta\mu}$ is reduced (just as problem A in refs. 10 and 11) to an "L-problem" (ref. 12) in the functional space $\{h_1(\tau),\ldots,h_n(\tau)\}$ with a norm

$$\|h\| = \int_0^T \left[\sum_{i=1}^n h_i^2(\tau)\right]^{\frac{1}{2}} d\tau,$$

then, as in ref. 11, we arrive at the following statement: problem $A_{\Theta\mu}$ has a single solution; the optimal time $T^0(\Theta,\mu)$ is the least root of the equation

$$\lambda(T, \Theta, \mu) = \min \int_0^T \left[\sum_{j=1}^n \left[\sum_{i=1}^n \lambda_i h_{ij}(\tau, \Theta, \mu)\right]^2\right]^{\frac{1}{2}} d\tau = 1 \qquad (2.4)$$

$$\left(\sum_{i=1}^n c_i \lambda_i = 1\right),$$

where c_i are components of vector (2.2) $(0 \leqslant \Theta \leqslant 1)$, while the optimal control is defined by the formula

$$u_j^0(t, \Theta, \mu) = \frac{\sum_{i=1}^n \lambda_i^0 h_{ij}(\,\cdot\,, \Theta, \mu)}{\left[\sum_{j=1}^n \left[\sum_{i=1}^n \lambda_i^0 h_{ij}(t, \Theta, \mu)\right]^2\right]^{\frac{1}{2}}}, \quad (j=1,\ldots,n). \quad (2.5)$$

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*This formula, of course, also follows from L.S. Pontryagin's Maximum Principle (ref. 2).

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Here $\lambda_i^o(\theta, \mu)$ are the solutions to problem (2.4) and the denominator (2.5) does not go to zero because matrix $B(\mu)$ is non-singular.

Relationships (2.1)-(2.5) make it possible to establish the reducibility of the solution to problem $A_{1\mu}$ to that of A as $\mu \to 0$.

Lemma 2.1. The limit

$$\lim_{\mu \to 0} T^o(1, \mu) = T^o \qquad (2.6)$$

holds.

Proof. Since, obviously, $\lambda(T, 1, \mu) \geq \lambda(T)$, from (2.1) and (2.4) there follows the inequality

$$T^o(1, \mu) \leq T^o. \qquad (2.7)$$

The optimal control time of problem A (T^o) is the least root of equation (2.1); hence

$$\lambda(T) < 1 \text{ for } T \leq T^o - \alpha \qquad (\alpha > 0). \qquad (2.8)$$

On the other hand, for $0 \leq T \leq T^o - \alpha$ we have the uniform limit

$$\lim_{\mu \to 0} \lambda(T, 1, \mu) = \lambda(T), \qquad (2.9)$$

and, consequently, for sufficiently small values of μ, the inequality

$$\lambda(T, 1, \mu) < 1 \text{ for } T \in [0, T^o - \alpha].$$

In combination with inequality (2.7), the latter inequality means that the number $T^o(1, \mu)$ is the least root of equation (2.4) and lies on the segment $[T^o - \alpha, T^o]$ for sufficiently small $\mu > 0$. As a result of the arbitrary choice of $\alpha > 0$, the lemma is thus proved.

Lemma 2.2. The first component $u_1^o(t, 1, \mu)$ of the optimal control of problem $A_{1\mu}$ converges to the optimal control $\eta^o(t)$ of problem A.

Proof. The numbers $\lambda_i^o(1, \mu)$ which are the solutions to problem (2.4) for $\theta = 1$ are uniformly limited as $0 < \mu < \mu_o$ ($\mu_o > 0$ is a constant. Consequently, it is possible to choose a convergent subsequence $\{\lambda_i^o(1, \mu_{k_j})\}$ from any

sequence $\{ \lambda_i^o(1, \mu_k)\}$ (k=1,2,...), where $\lim\limits_{k\to\infty} \mu_k = 0$. Likewise, it is not difficult to ascertain that the numbers $\lambda_i^* = \lim\limits_{k\to\infty} \lambda_i^o(1, \mu_{k_j})$ are solutions to problem (2.1). Now the lemma is proved with the aid of formulas (2.3) and (2.5).

Note. The optimal trajectories which are defined by the least roots of equations (2.1) and ((2.4)) correspond to the minimum time T^o ($T^o(\theta, \mu)$) relative to various variations of controls $\eta^o(t)$ ($u^o(t)$) constrained by conditions (0.2) ((1.4)).

3. In the presence of externally impressed functions $\varphi(t) \neq 0$ in linear systems, the existence of locally-optimal trajectories is possible. Let us consider the situation which arises here.

Definition 3.1. Trajectory $x(x_0, t, \eta^o)$ of system (0.1) (trajectory $x(x_0, t, u^o)$ of system (1.1) respectively will be called <u>locally optimal</u> if it is possible to find a number $\xi > 0$ such that there does not exist any permissible control $\eta(t)$ ($u(t)$) subject to conditions (0.2) (1.4) such that the trajectory it generates $x(x_0, t, \eta)$ ($x(x_0, t, u)$ reaches point $x = 0$ for $t = T < T^o (T < T^o, \theta, \mu)$), with the added condition that the inequality

$$\sum_{i=1}^{n} [x_i(x_0, t, \eta) - x_i(x_0, t, \eta^o)]^2 \leq \epsilon^2 \quad (0 \leq t \leq T)$$

(3.1)

(or, respectively, the inequality)

$$\sum_{i=1}^{n} [x_i(x_0, t, u) - x_i(x_0, t, u^o)]^2 \leq \epsilon^2 \quad (0 \leq t \leq T)$$

(3.2)

is fulfilled.

Note. In order to establish the local optimality of the trajectory $x(x_0, t, \eta^o)$ ($x(x_0, t, u^o)$ in our case, it is sufficient to prove the existence of a number $\delta > 0$ such that there exists no permissible control $\eta(t)$ ($u(t)$) which will

bring trajectory $x(x_0, t, \eta)$ $(x(x_0, t, u))$ to point $x = 0$
at moment $t = T$ $(T^0 - \delta, T^0)$.

Lemma 3.1. If T^0 $(T^0(\theta, \mu))$ is a root of equation
(2.1) ((2.4)), and the inequality

$$\lambda(T) < 1 \text{ for } T \in (T^0 - \alpha, T^0) \quad (\alpha > 0) \qquad (3.3)$$

(or $\quad \lambda(T, \theta, \mu) < 1 \text{ for } T \in (T^0(\theta, \mu) - \alpha, T^0(\theta, \mu)$

$$(\alpha > 0) \quad (3.4)$$

respectively) holds, then trajectory $x(x_0, t, \eta^0)$ $(x(x_0, t, u^0))$
generated by control (2.3) ((2.5)) is locally optimal.

Proof. To prove this lemma, it is sufficient to note
that in view of the general results relating to the "L-problem"
(ref. 12), under conditions (3.3)((3.4)) there exists no
permissible function $\eta(t)$ $(u(t))$ which satisfies inequality
(0.2) ((1.4)) and the condition

$$x(x_0, T, \eta) = F(T, 1)x_0 + \int_0^T F(T, 1)F^{-1}(\tau, 1)[b\eta(\tau) + \varphi(\tau)]d\tau = 0$$

(or condition

$$x(x_0, T, u) = F(T, \theta)x_0 + \int_0^T F(T, \theta)F^{-1}(\tau, \theta)[Bu(\tau) + \theta\varphi(\tau)]d\tau = 0$$

respectively).

Lemma 3.2. If T^0 is a root of equation (2.1) and

$$\frac{d\lambda}{dT} > 0 \quad \text{for} \quad T = T^0$$

then for $\mu \in (0, \mu_0)$ there exists a family of locally optimal
trajectories $x(x_0, t, u^0)_\mu$ of problem $A_{\lambda, \mu}$ which converge as
$\mu \to 0$ to the locally optimal trajectory $x(x_0, t, \eta^0)$ of problem A
furthermore,

$$\lim_{\mu \to 0} T^0(1, \mu) = T^0$$

and the first component $u_z^0(t, 1, \mu)$ converges uniformly to
function $\eta^0(t)$ as $\mu \to 0$.

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Lemma 3.3. Let $T^o(1, \mu)$ be a root of equation (2.4) for $\mu \in (0, \mu_o)$, $\Theta = 1$, continuously dependent on μ; in addition, let the following conditions be fulfilled:

$$\frac{\partial \lambda(T, 1, \mu)}{\partial T} > \alpha > 0 \quad (\alpha = \text{const}) \qquad (3.6)$$

$$T^o(1, \mu) < N < \infty \quad (N = \text{const}) \qquad (3.7)$$

Then problem A has a locally optimal trajectory $x(x_o, t, \eta^o)$ to which corresponds optimal control time

$$T^o = \lim_{\mu \to 0} T^o(1, \mu) \qquad (3.8)$$

and the optimal control function

$$\eta^o(t) = \text{sign}\left[\sum_{i=1}^{n} \lambda_i h_{i1}(t)\right],$$

$$\lambda_i = \lim_{\mu \to 0} \lambda_i^o(1, \mu). \qquad (3.9)$$

The proof of lemmas 3.2 and 3.3 makes use of arguments which are analogous to those applied in proving lemmas 2.1 and 2.2. For this reason, we omit the proofs of the former two lemmas here.

Note 1. Let us point out that conditions (2.1)-(2.5) which determine the solutions to problems A and $A_{\Theta\mu}$ are analogous to conditions which may be deduced from the maximum principle for stationary systems given in ref. 2. The form used in (2.1)-(2.5) is convenient for what follows.

Note 2. In case $\varphi \equiv 0$ there exist no locally optimal trajectories which differ from the single optimal trajectory $x(x_o, t, \eta^o)$, since equation (2.1)((2.4)) cannot have more than one root (when $\varphi \equiv 0$, $\mu \neq 0$, and $\partial \lambda / \partial T > 0$ everywhere).

4. Lemmas 2.1-3.3 point to the following method of constructing an optimal trajectory for system (0.1):

We construct the auxiliary system (1.1); for $\Theta = 0$, $\mu = 1$ the optimal trajectory of system (1.1) which connects points $x = x_o$ and $x = 0$ is found by an elementary method: this is the straight line which connects points $x = x_o$, $x = 0$. Let us have a solution to the problem for some value $\Theta \geq 0$ ($\mu = 1$); then, using the theorems on implicit functions

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[See note*], it is possible to find expressions for derivatives $\frac{dT^o}{d\theta}$, $\frac{d\lambda_i^o}{d\theta}$ (provided that for these values of θ and μ

the partial derivative $\frac{\partial\lambda(T,\theta,\mu)}{\partial T}$ is different from zero).

Integration of the resultant differential equations makes it possible to extend the solution to the optimal problem onto the next larger values of θ. On reaching point $\theta = 1$, the next step is to construct by analogous means differential equations containing derivatives $dT^o/d\mu$ and $d\lambda_i^o/d\mu$ and to integrate these equations for $\mu\in(0,1]$. It is also possible to vary θ and μ concurrently, by setting $\mu = 1 - \theta$, for example.

Let us construct the above-mentioned differential equations. If for certain values of θ, μ there holds the inequality

$$\frac{\partial\lambda(t,\theta,\mu)}{\partial T} \neq 0, \tag{4.1}$$

then, according to the known theorems on implicit functions, we have

$$\frac{dT^o}{d\theta} = -\frac{\partial\lambda/\partial\theta}{\partial\lambda/\partial T} \tag{4.2}$$

where the expression $\lambda(T,\theta,\mu)$ is defined by equation (2.4).

To determine the values of $\lambda_i^o(\theta,\mu)$ which solve problem (2.4) for its minimum for a fixed value of T, it is possible to make use of the rule for Lagrange multipliers or to express one of the numbers λ_i (let us say λ_1) in terms of the other numbers λ_j by means of the condition [see note**]

$$\sum_{i=1}^{n}\lambda_i c_i = 1. \tag{4.3}$$

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* All functions in equation (2.4) are continuously differentiable functions of their arguments (for $x \neq 0$). The introduction of parameter μ is intended to assure this differentiability.

**Among the numbers c_i there is at least one which differs from zero. If all c_i's turn out to be identically zero, this means that the trajectory reaches point $x = 0$ for $u = 0$, and so for such c_i's, time T^o cannot be the optimal control time. In changing parameters θ and μ, the number i for which $c_i \neq 0$, can, generally speaking, undergo alteration.

let us consider the latter method.

We shall introduce the expression:

$$\gamma[T, \theta, \mu, \{\lambda_i\}] = \int_0^T \left[\sum_{j=1}^n \left(\sum_{i=2}^n \lambda_i h_{ij} + \frac{1}{c_1} \left(1 - \sum_{k=2}^n c_k \lambda_k \right)^2 \right] \right]^{1/2} d\tau. \quad (4.4)$$

Then to determine the numbers λ_i^o we solve the system of equations

$$\frac{\partial \gamma}{\partial \lambda_i} = 0 \quad (i = 2, \ldots, n). \quad (4.5)$$

The functional determinant of the right members of system (4.5) with respect to variables λ_i satisfies the condition.

$$\frac{D\left[\frac{\partial \gamma}{\partial \lambda_2}, \ldots, \frac{\partial \gamma}{\partial \lambda_n} \right]}{D[\lambda_2, \ldots, \lambda_n]} \neq 0. \quad (4.6)$$

To prove relationship (4.6) it is sufficient to establish that the quadratic form

$$w(z) = \sum_{i,j=2}^n \frac{\partial^2 \gamma}{\partial \lambda_i \partial \lambda_j} z_i z_j$$

is positive definite. The analytic confirmation of this latter fact, readily apparent from geometric considerations, will be omitted here.

Consequently, derivatives $d\lambda_i^o/d\theta$ satisfy the equations

$$\frac{d\lambda_i^o}{d\theta} = - \frac{\dfrac{D\left[\frac{\partial \gamma}{\partial \lambda_2}, \ldots, \frac{\partial \gamma}{\partial \lambda_n} \right]}{D[\lambda_2, \ldots, \theta, \ldots, \lambda_n]}}{\dfrac{D\left[\frac{\partial \gamma}{\partial \lambda_2}, \ldots, \frac{\partial \gamma}{\partial \lambda_n} \right]}{D[\lambda_2, \ldots, \lambda_n]}}. \quad (4.7)$$

$$(i = 2, \ldots, n)$$

Equations analogous to (4.2) and (4.7) can also be constructed for derivatives $dT^o/d\mu$, $d\lambda_i^o/d\mu$:

$$\frac{dT^o}{d\mu} = -\frac{\frac{\partial \gamma}{\partial \mu}}{\frac{\partial \gamma}{\partial T}} \qquad (\theta = 1), \qquad (4.8)$$

$$\frac{d\lambda_i^o}{d\mu} = -\frac{\dfrac{D\left[\frac{\partial \gamma}{\partial \lambda_2}, \cdots, \frac{\partial \gamma}{\partial \lambda_n}\right]}{D\left[\lambda_2, \cdots, \mu, \cdots \lambda_n\right]}}{\dfrac{D\left[\frac{\partial \gamma}{\partial \lambda_2}, \cdots, \frac{\partial \gamma}{\partial \lambda_n}\right]}{D\left[\lambda_2, \cdots, \lambda_n\right]}} \qquad (4.9)$$

$$(i = 2,\ldots,n; \ \theta = 1).$$

Note. In calculating the partial derivatives $\partial \lambda / \partial T$, cognizance should be taken of the complex nature of the dependence of $\lambda(T, \theta, \mu)$ on T; in the first place, on the upper limit of the integral, and secondly, through the c_i's. Analogously, in calculating the functional determinants in the numerators of the right members of (4.7) and (4.9), it should be borne in mind that in deriving equations (4.7)-(4.8) the expression $T^o(\theta, \mu)$ was assumed to be a known function of parameters $\theta \ \mu$. In case $\mu = 1 - \theta$, it is sufficient to take into consideration only equations (4.2), (4.7); in doing this, however, the dependence of μ on θ should be taken into account in the functional determinant in the numerator of the right member of (4.7).

A consequence of lemmas 2.1-3.3 is the following

Theorem 4.1. Let $T^o(\theta, \mu), \lambda_i^o(\theta, \mu)$ $(0 \le \theta < 1; \ 0 < \mu \le 1, \ \theta = 1)$ be solutions to the system of equations (4.2), (4.7), (4.8), (4.9) and the condition

$$\partial \lambda / \partial T > \alpha > 0 \quad (\alpha = \text{const}) \qquad (4.10)$$

be satisfied for these solutions.

Then the number T^o defined by the limit

$$T^o = \lim_{\mu \to 0} T^o(1, \mu), \qquad (4.1\ )$$

is the optimal control time of problem A; time T^o has a corresponding locally optimal trajectory with optimal control

$$\eta^o(t) = \text{sign}\left[\sum_{i=1}^{n} \lambda_i b_{1i}(t)\right], \ \lambda_i = \lim_{\mu \to 0} \lambda_i^o(1, \mu). \qquad (4.12$$

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Note. In case $\varphi \equiv 0$, under conditions (1.7), the integration of system (4.2), (4.7)-(4.9) always leads to a single optimal trajectory $x(x_0, t, \eta^0)$.

5. In this paragraph we shall consider problem B.

Let us replace problem B with problem $B_{\theta\mu}$ for the auxiliary system (1.1). To derive the equations describing the dependence of optimal solutions $T^0(\theta, \mu)$, $\lambda^0(\theta, \mu)$ of problem $B_{\theta\mu}$ on parameters θ, μ, it is necessary to repeat the construction of equations (4.2), (4.7)-(4.9) of section 4. To these equations, however, in which the c_i's are calculated from the formulas [see note*]

$$c_i = -x_{10} - \int_0^T \{F^{-1}(\tau, \theta) \, \varphi(\tau)\}_i \theta \, d\tau + \{F^{-1}(T, \theta)\xi^0\}_i$$

(5.1)

$$(i = 1, \ldots, n; \text{ vector } \xi^0 \in M)$$

we should add the equations which describe the changes of coordinates $\xi_i^0(\theta, \mu)$ of point $\xi^0 \in M$ with alterations in parameters θ and μ.

Let manifold M be described by the continuously differentiable functions

$$\xi_i = \xi_i(\zeta_1, \ldots, \zeta_r) \quad (i = 1, \ldots, n).$$

(5.2)

The extremality conditions for time $T^0(\theta, \mu)$ yield the system of equations

$$\partial T / \partial \zeta_i = 0 \quad (i = 1, \ldots, r),$$

(5.3)

which is satisfied by the variables ζ_i^0 $(i = 1, \ldots, r)$ that define coordinates ξ_j^0 of extremal point $\xi^0 \in M$.

If for the examined values of θ and μ the functional determinant of the left members of (5.3) in terms of variables ζ_i is not equal to zero, then derivatives $d\zeta_i / d\theta$ (for $0 \leq \theta \leq 1$, $\mu = 1$) and $d\zeta_i / d\mu$ (for $\theta = 1$, $0 < \mu \leq 1$) satisfy the conditions

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*Index i in equation (5.1) stands for the i-th component of the corresponding vector.

$$\frac{d\xi_i}{d\theta} = -\frac{\dfrac{D\left[\dfrac{\partial T}{\partial \xi_1}, \ldots, \dfrac{\partial T}{\partial \xi_r}\right]}{D[\xi_1, \ldots, \theta, \ldots, \xi_r]}}{\dfrac{D\left[\dfrac{\partial T}{\partial \xi_1}, \ldots, \dfrac{\partial T}{\partial \xi_r}\right]}{D[\xi_1, \ldots, \xi_r]}} \qquad (i = 1, \ldots, r) \qquad (5.4)$$

$$\frac{d\xi_i}{d\mu} = -\frac{\dfrac{D\left[\dfrac{\partial T}{\partial \xi_1}, \ldots, \dfrac{\partial T}{\partial \xi_r}\right]}{D[\xi_1, \ldots, \mu, \ldots, \xi_r]}}{\dfrac{D\left[\dfrac{\partial T}{\partial \xi_1}, \ldots, \dfrac{\partial T}{\partial \xi_r}\right]}{D[\xi_1, \ldots, \xi_r]}} \qquad (i = 1, \ldots, r), \qquad (5.5)$$

where derivatives $\partial T/\partial \xi_i$ should be calculated from equation (2.4) by means of the rule for differentiating implicit functions. System of equations (5.4), (5.5) in combination with equations (4.2), (4.7)-(4.9) describes the dependence of the optimal solutions to problem $B_{\theta\mu}$ on parameters θ and μ.

The truth of the following statement is established by combining lemmas 3.1-3.3 with sufficient conditions to make a function of r variables $\xi_1, \ldots, \xi_r$ assume its minimum value by the usual methods employed in the calculus of variations.

If $T^0(\theta, \mu)$, $\lambda_i^0(\theta, \mu)$, $\xi_k^0(\theta, \mu)$ are solutions

to the system of equations (4.2), (4.7)-(4.9), (5.4), (5.5), along which inequalities (4.10) and

$$\frac{D\left[\dfrac{\partial T}{\partial \xi_1}, \ldots, \dfrac{\partial T}{\partial \xi_r}\right]}{D[\xi_1, \ldots, \xi_r]} \neq 0,$$

and for sufficiently small values of $\mu \, (0 < \mu < \mu_0)$ the quadratic form

$$v(z) = \sum_{i,j=1}^{r} \frac{\partial^2 T}{\partial \xi_i \partial \xi_j} z_i z_j$$

is negative definite uniformly with respect to μ (i.e., $v(z) \geq \alpha \sum z_i^2$ ($\alpha > 0$ is a constant), then

$$T^o = \lim_{\mu \to 0} T^o(1, \mu)$$

is the optimal control time for problem B, which has a corresponding locally optimal trajectory $x(x_o, t, \eta^o, M)$ generated by the control

$$\eta^o(t) = \text{sign} \left[\sum_{i=1}^{n} \lambda_i h_{i1}(t) \right],$$

where the numbers λ_i are defined by the limit

$$\lambda_i = \lim_{\mu \to 0} \lambda_i^o(1, \mu).$$

Note. System (4.2), (4.7)-(4.9), (5.4)-(5.5) is one of the possible forms of the system of equations which describe the dependence of the solutions to problems $A_{\theta\mu}$ and $B_{\theta\mu}$ on θ and μ. Cases may occur when in integrating this system, the combination of $n-1$ indices i should be changed in equations (4.7),(4.9). Sometimes in deriving such equations, instead of directly replacing variable λ_i with an expression in terms of the rest of the variables λ_j, it is more advisable to make use of the Lagrange multiplier method and to replace the two systems (4.2), (4.7), (4.8), (4.9) by a single system (4.2), (4.7), setting $\mu = 1 - \theta$.

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